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On the Swirling Flow Between Rotating Coaxial Disks

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INTRODUCTION

We consider here a system of ordinary differential equations describing the steady flow of a Navier-Stokes fluid contained between two parallel, infinite, plane disks which are rotating about a common axis with constant angular velocities. The idea of seeking similarity solutions of the Navier-Stokes equations for the flow above a single rotating disk goes back to Von Kárman [1], and the existence problem for that situation has been extensively studied [8–12]. Batchelor [2], Stewartson [3], Lance and Rodgers [4], and Pearson [5] have studied the flow between two disks using a variety of numerical and approximate methods. Their conclusions have differed somewhat, especially when the angular velocities are equal and opposite and the Reynolds number is large. On the other hand, there has been no rigorous proof of the existence of a solution of the differential equations until recently, when Hastings [6] proved that a solution exists if the angular velocities of both disks are close to zero. The interesting case in which the angular velocities are equal and opposite has been dealt with by McLeod and Parter [7] in a recent work. They prove existence, for all values of the angular velocity, of a solution with the symmetry that the geometry suggests. Furthermore, they obtain detailed information about the asymptotic behavior of solutions of this type for large Reynolds numbers. However, their methods rely heavily on symmetry considerations, and it is not clear how to extend their results to other cases. In what follows we will prove existence of a solution if (i) the larger angular velocity is less than a constant C, roughly equal to 1.5, or if (ii) the disks rotate with the same sense and the difference of the squares of the angular velocities is less than C^2. This may be thought of as a finite perturbation from the trivial (rigid motion) solution which occurs when the angular velocities are equal.

The situation may be indicated by a diagram (Fig. 1). The lightly shaded area is the domain of values (Ω_0, Ω_1) in which we prove existence. The line $\Omega_0 = -\Omega_1$ represents the case dealt with by McLeod and Parter, and the square about the origin (of undetermined size) corresponds to the work of Hastings.

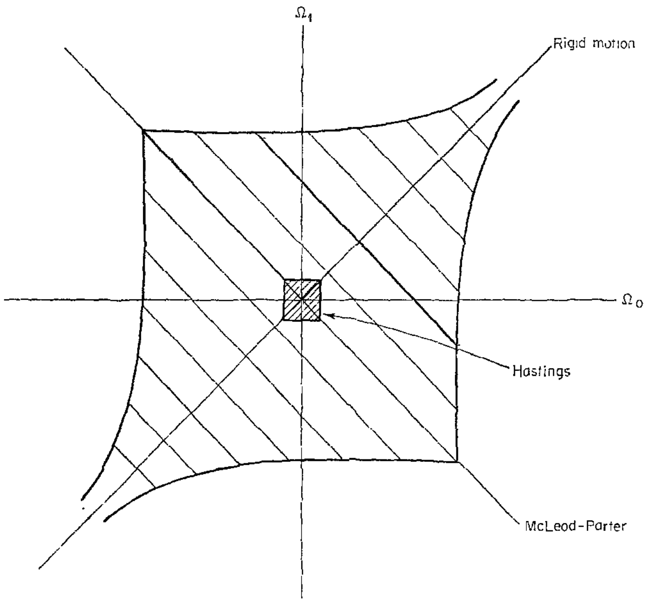

FIGURE 1

Suppose that cylindrical coordinates (r, θ, z) are introduced, and that the corresponding velocity components are (u, v, w). If we substitute

$$u = rf'(z), \qquad v = rh(z), \qquad w = -2f(z)$$

in the Navier-Stokes equations, finding a solution (u, v, w) of our flow problem is equivalent to finding a real number μ and functions f, h such that

$$\nu f''' + 2ff'' + (h^2 - f'^2) = \mu$$
$$\nu h'' + 2fh' - 2f'h = 0$$
$$f(0) = f'(0) = f(d) = f'(d) = 0, \qquad h(0) = \tilde{\Omega}_0, \qquad h(d) = \tilde{\Omega}_1$$

where $\tilde{\Omega}_0$ and $\tilde{\Omega}_1$ are the angular velocities of the disks located at $z = 0$ and $z = d$, respectively. We may assume without loss of generality that $|\tilde{\Omega}_1| \leqslant \tilde{\Omega}_0$. With the changes of variables

$$z \to d\eta, \quad f \to \frac{\nu}{2d} f, \quad h \to \frac{\nu}{2d^2} h,$$

and after differentiation of the first differential equation, our problem becomes

$$f'''' + ff''' + hh' = 0$$

$$h'' + fh' - f'h = 0$$

$$f(0) = f'(0) = f(1) = f'(1) = 0,\ h(0) = \frac{2d^2\tilde{\Omega}_0}{\nu} \equiv \Omega_0\,,\ h(1) = \frac{2d^2\tilde{\Omega}_1}{\nu} \equiv \Omega_1\,.$$

This is the boundary value problem which is studied below.

We will need some preliminary results, stated as lemmas, before giving the proof of our main theorem.

LEMMA 1. *Suppose that* $g \in C^2[0, 1]$, $g(0) = g'(0) = g(1) = g'(1) = 0$, *and that* $|g''| \leqslant a$. *Then*

(a) $|g'| \leqslant \min\{a\eta, a(1-\eta)\}$,

(b) $|g| \leqslant \varphi'(\eta)$,

(c) $|\int_0^\eta g| \leqslant \varphi(\eta)$,

where

$$\varphi(\eta) = \begin{cases} \dfrac{a\eta^3}{6} & \eta \leqslant \frac{1}{2} \\[2mm] \dfrac{a\eta^3}{6} + \dfrac{a}{2}\eta(1-\eta) - \dfrac{a}{8}\eta & \eta > \frac{1}{2}\,. \end{cases}$$

The proof is an elementary application of the fundamental theorem of calculus and the boundary conditions on g. It is important for us to note that φ is increasing on $[0, 1]$.

LEMMA 2. *Suppose that* $g \in C^2[0, 1]$, $g(0) = g'(0) = g(1) = g'(1) = 0$, *and that* $|g''| < 8/3$. *Then there exists a unique solution of the problem*

$$Lh = h'' + gh' - g'h = 0, \qquad h(0) = \Omega_0\,, \qquad h(1) = \Omega_1\,. \tag{1}$$

Furthermore, this solution is nonincreasing on $[0, 1]$.

Proof. We define $H = h/w_1$, where

$$w_1 = 1 - \beta\eta^2,$$

and $\beta \in (0, 1)$ is to be chosen. The equation $Lh = 0$ is equivalent to

$$H'' + \left(\frac{2w_1'}{w_1} + g\right) H' + \left(\frac{Lw_1}{w_1}\right) H = 0,$$

and, if $Lw_1 \leqslant 0$, the maximum principle ([13, Chapter 1]) implies that H cannot have a positive maximum or a negative minimum in (0, 1). The assumption that $a = \max | g'' |$ is smaller than 8/3, together with the bounds on g and g' given in Lemma 1, implies that $Lw_1 \leqslant 0$ if we choose

$$\beta = 2a/8 - a.$$

Uniqueness for the problem $Lh = 0$, $h(0) = h(1) = 0$ implies existence of a unique solution for (1), so the first assertion of the lemma is proven.

The proof that h is decreasing is separated into two cases.

First suppose that $\Omega_1 \in [-\Omega_0, 0]$. We define $K = h/w_2$ where

$$w_2 = 1 - \beta(1 - \eta)^2.$$

The assumptions on g imply that $Lw_2 \leqslant 0$, so that K satisfies the maximum principle on (0, 1). Since H and K can have neither a positive maximum nor a negative minimum in (0, 1), and since both are positive at 0 and non-positive at 1, it follows that both are nonincreasing on [0, 1]. We also observe that H, K, h are positive, zero, and negative together, and that there is a unique $\eta_0 \in (0, 1)$ such that $h(\eta_0) = 0$. It follows from $H' \leqslant 0$, that $h' \leqslant -2\beta\eta H$, so that $h' \leqslant 0$ on $[0, \eta_0]$. Similarly $K' \leqslant 0$ implies that $h' \leqslant 2\beta(1 - \eta)K$ so that $h' \leqslant 0$ on $[\eta_0, 1]$.

Now consider the case $\Omega_1 \in (0, \Omega_0]$. First observe that the maximum principle implies that h is positive [0, 1] and $h'(0)$ is not positive. We define $M = h/w_3$ where

$$w_3 = 1 + \gamma\eta^2,$$

$\gamma > 0$ to be determined, and $R = M'/M$. Then

$$R' + R^2 + \left(\frac{2w_3'}{w_3} + g\right) R + \left(\frac{Lw_3}{w_3}\right) = 0,$$

and, since

$$M' = \frac{h'}{w_3} - \frac{hw_3'}{w_3^2},$$

$R(0) = h'(0)/h(0) \leqslant 0$. Another application of the bounds for g and g' given in Lemma 1 implies that if γ is appropriately chosen, for example $\gamma = 2$, that

$$Lw_3 \geqslant 0 \qquad \text{and} \qquad 2w_3'/w_3 + g \geqslant 0.$$

It follows that $R \leqslant 0$, and since $M > 0$, that $H' \leqslant 0$, so that

$$h' \leqslant -hw_3'/w_3 \leqslant 0.$$

The proof is completed.

LEMMA 3. *Suppose that* $g \in C[0, 1]$, *and* $h \in C'[0, 1]$. *Then there is a unique solution of*

$$f'''' + gf'' + hh' = 0, \qquad f(0) = f'(0) = f(1) = f'(1) = 0. \tag{2}$$

Proof. The differential equation can be integrated once, yielding

$$f'''(\eta) = -\tfrac{1}{2}\exp\left(-\int_0^\eta g\right)\int_0^\eta \exp\left(\int_0^s g\right) d[h^2(s)] + k\exp\left(-\int_0^\eta g\right) \equiv F(\eta), \tag{3}$$

where k is a constant to be determined. This equation together with $f'(0) = f'(1) = 0$ is equivalent to

$$f'(\eta) = \int_0^1 g(\eta, \xi) F(\xi)\, d\xi, \tag{4}$$

where

$$G(\xi, \eta) = \begin{cases} (1-\xi)\eta & \eta \leqslant \xi \\ (1-\eta)\xi & \eta > \xi. \end{cases}$$

If f is defined to be the integral of its derivative, it remains to satisfy the boundary condition $f(1) = 0$, which is equivalent to

$$k\int_0^1 \xi(1-\xi)\exp\left(-\int_0^\xi g\right) d\xi$$
$$= \tfrac{1}{2}\int_0^1 \xi(1-\xi)\exp\left(-\int_0^\xi g\right)\int_0^\xi \exp\left(\int_0^s g\right) d[h^2(s)]\, d\xi. \tag{5}$$

This expression for k will be essential in what follows. For brevity in the statement of our main theorem we define

$$C = \tfrac{4}{3}(2)^{1/2}\,(e^{1/4} + e^{4/9} - 1)^{-1/2} \doteq 1.5.$$

THEOREM. *There exists a solution of the system*

$$f'''' + ff''' + hh' = 0, \qquad f(0) = f'(0) = f(1) = f'(1) = 0,$$
$$h'' + fh' - f'h = 0, \qquad h(0) = \Omega_0, \qquad h(1) = \Omega_1,$$

provided either

(a) $\Omega_1 \in [-\Omega_0, 0]$ *and* $0 \leqslant \Omega_0 < C$, *or*

(b) $\Omega_1 \in (0, \Omega_0]$ *and* $\Omega_0^2 - \Omega_1^2 < C^2$.

This result may be thought of as showing that there is a solution if the prescribed angular velocities are sufficiently close either to zero or to the case $\Omega_1 = \Omega_0$ in which there is a trivial solution $h \equiv \Omega_0$, $f \equiv 0$ (rigid motion).

Proof. We define

$$S_a = \{g \in C^2[0, 1] \mid |g''| \leqslant a, g(0) = g'(0) = g(1) = g'(1) = 0\}.$$

For $a < 8/3$, we define the mapping T of S_a into $C^2[0, 1]$ by $f = T(g)$, where h is the solution of (1) and f the solution of (2) for this h.

The bulk of the work is in determining whether T maps S_a into itself. Suppose that $g \in S_a$, and that F is given by (3) and k by (5). It follows from (4) that

$$f''(\eta) = -\int_0^1 \xi F(\xi)\, d\xi + \int_\eta^1 F(\xi)\, d\xi,$$

and if we derive an estimate $|F| \leqslant b$, it then follows that $|f''| \leqslant \frac{3}{2}b$. Therefore, if $b \leqslant \frac{2}{3}a$, T maps S_a into itself.

We will deal with case (b) first. The distinguishing feature of this case is that h^2 is nonincreasing.

This implies that

$$[\Omega_0^2 - h^2(\eta)]\, e^{-\varphi(\eta)} \leqslant \int_0^\eta \exp\left(\int_0^s g\right) d[-h^2(s)] \leqslant [\Omega_0^2 - h^2(\eta)]\, e^{\varphi(\eta)}.$$

We have invoked Lemma 1 and the fact that φ is increasing here. It then follows that

$$F_1 \leqslant F \leqslant F_2,$$

where

$$F_2(\eta) = \tfrac{1}{2}[\Omega_0^2 - h^2(\eta)]\, e^{2\varphi(\eta)},$$

and

$$F_1(\eta) = -\tfrac{1}{2}\, e^{\varphi(\eta)} \int_0^1 \xi(1-\xi)\, e^{2\varphi(\xi)}[\Omega_0^2 - h^2(\xi)]\, d\xi \left(\int_0^1 \xi(1-\xi)\, e^{-\varphi(\xi)}\, d\xi\right)^{-1}.$$

This implies that

$$-\tfrac{1}{2}(\Omega_0^2 - \Omega_1^2)\, e^{a/6} \leqslant F(\eta) \leqslant \tfrac{1}{2}(\Omega_0^2 - \Omega_1^2)\, e^{a/12},$$

and that

$$|F(\eta)| \leqslant \tfrac{1}{2}(\Omega_0^2 - \Omega_1^2)\, e^{a/12}.$$

Therefore, if

$$(\Omega_0^2 - \Omega_1^2) \leqslant \tfrac{4}{3} a e^{-a/12} \equiv \psi(a),$$

T maps S_a into itself. Since ψ is increasing on $(0, 8/3)$, if

$$\Omega_0^2 - \Omega_1^2 < \psi\left(\frac{8}{3}\right) = \frac{32}{9} e^{-2/9},$$

T maps S_a into itself if a is chosen sufficiently close to $8/3$.

We return to case (a). An integration by parts yields

$$\exp\left(\int_0^\eta g\right) F(\eta) = k + \tfrac{1}{2}\Omega_0^2 - \tfrac{1}{2}\exp\left(\int_0^\eta g\right) h^2(\eta) + \tfrac{1}{2}\int_0^\eta h^2(s)\exp\left(\int_0^s g\right) g(s)ds$$
$$\equiv k + q(\eta).$$

We have

$$2q(\eta) \leqslant \Omega_0^2 - h^2(\eta)\, e^{-\varphi(\eta)} + \int_0^\eta h^2(s)\, e^{\varphi(s)}\, \varphi'(s)\, ds \leqslant \Omega_0^2\, e^{\varphi(\eta)},$$

and

$$2q(\eta) \geqslant \Omega_0^2 - h^2(\eta)\, e^{\varphi(\eta)} - \int_0^\eta h^2(s)\, e^{-\varphi(s)}\, \varphi'(s)\, ds \geqslant -\,\Omega_0^2[e^{\varphi(\eta)} - e^{-\varphi(\eta)}].$$

Also, then

$$k \geqslant -\frac{\Omega_0^2}{2}\int_0^1 (1-\xi)\, e^{2\varphi(\xi)}\, d\xi \cdot \left(\int_0^1 \xi(1-\xi)\, e^{-\varphi(\xi)}\, d\xi\right)^{-1} \geqslant -\frac{\Omega_0^2}{2}\, e^{3a/24}$$

and

$$k \leqslant \frac{\Omega_0^2}{2}\int_0^1 \xi(1-\xi)[e^{2\varphi(\xi)} - 1]\, d\xi \cdot \left(\int_0^1 \xi(1-\xi)\, e^{-\varphi(\xi)}\, d\xi\right)^{-1}$$
$$\leqslant \frac{\Omega_0^2}{2}\,[e^{3a/24} - e^{a/24}].$$

Then

$$\frac{\Omega_0^2}{2}\,[1 - e^{a/12} - e^{a/6}] \leqslant F \leqslant \frac{\Omega_0^2}{2}\, e^{a/6},$$

and

$$|F| \leqslant \frac{\Omega_0^2}{2}\,[e^{a/6} + e^{a/12} - 1].$$

As before, if $\Omega_0 < C$, and a is chosen sufficiently close to 8/3, T maps S_a into itself.

In order to complete the proof we must show that T is continuous and maps S_a into a compact part of itself. The compactness of $T(S_a)$ follows immediately from the Arzela-Ascoli theorem, since for $g \in S_a$, $|T(g)'''(\eta)| \leqslant \frac{2}{3}a$. The continuity of T follows from continuity of solutions of (1) with respect to g, which follows from the uniformity of convergence of Picard iterations to a solution of (1) for $g \in S_a$. Q.E.D.

Remark. The value of C may be taken to be slightly larger in case (a), as is clear from the proof.

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